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A PECULIAR EXAMPLE IN MINIMA OF SURFACES

BY E. R. HEDRICK

THE theory of maxima and minima of a surface abounds in paradoxes of an instructive nature. One such, first given by Peano,* is now commonly cited: †

The surface

$$z = (y - x^2)(y - 2x^2) \tag{1}$$

has neither a maximum nor a minimum at the origin; yet every plane section of this surface by a plane through the z axis has an isolated proper minimum at the origin.

This is explained readily by a figure, which is essentially the same as that suggested below. It is evident that *cylinders* may be drawn which cut this surface in curves which do not have minima at the origin; for example, the circular cylinder $x^2 + (y - 1/3)^2 = 1/9$ gives a curve of intersection which actually has a maximum at the origin.

An example of even more curious behavior is the following:

The surface

$$\begin{aligned} z &= (y - e^{-\frac{1}{x^2}})(y - 2e^{-\frac{1}{x^2}}) \quad &\text{when } x \neq 0 \\ z &= y^2 \quad &\text{when } x = 0 \end{aligned} \tag{2}$$

is continuous, and z possesses finite partial derivatives of every order at every point. The surface has neither a maximum nor a minimum at the origin, yet every section by a right cylinder whose trace upon the xy plane

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* A. Genocchi and G. Peano, *Calcolo differenziale e principii di calcolo integrale*, 1884, p. XXIX; German translation by Bohlmann and Schepp, 1899, p. 332. Cf. p. III of the German translation.

† See, e. g., Goursat, *Course in Mathematical Analysis* (English trans.), vol. 1, p. 124. A similar property with reference to the distance from a point to a surface will be given by the writer in a paper to appear in the *Bulletin of the American Mathematical Society*.

is an analytic curve through the origin has an isolated proper minimum at the origin.[*]

The function $f(x) = e^{-\frac{1}{x^2}}$ (defined $= 0$ for $x = 0$), was first used by Cauchy, and is now a standard example of a curious function of a single variable;[†] $f(x)$, together with all its derivatives, is defined and continuous for all values of x; it is positive for $x \neq 0$; but all its derivatives vanish at $x = 0$, as does $f(x)$ itself. It follows that $f(x)$ is not expansible in Taylor's infinite series at $x = 0$.

Returning to the example (2) itself, it is seen that of the two factors of z, the first is positive for $y > e^{-\frac{1}{x^2}}$, the second for $y < e^{-\frac{1}{x^2}}$; hence if the two curves

$$C_1 : \quad y = e^{-\frac{1}{x^2}},$$

$$C_2 : \quad y = 2e^{-\frac{1}{x^2}},$$

be drawn, z is negative between C_1 and C_2, is zero on C_1 and on C_2, and otherwise is positive.

It is now easy to see that any analytic curve K through (0, 0) in the xy plane cannot penetrate the region between C_1 and C_2 inside of a sufficiently small circle about (0, 0).

First, suppose that K is not tangent to the x axis at (0,0); then evidently K has only the one point (0, 0) in common with C_1 and C_2 and otherwise lies wholly outside the region between the two, near the origin. Secondly, along the x axis itself, z is evidently positive except at the origin.

Finally, if K is tangent to the x axis at the origin, but does not coincide with it, then K has contact of some finite order with the x axis at (0, 0); hence the ordinate of K is negative or else it is greater than the corresponding ordinate of $y = 2e^{-\frac{1}{x^2}}$ for every value of x different from zero, near the origin, since C_2 itself has contact of infinitely high order with the x axis. The formal proof of this statement is easy, using Taylor's theorem with a remainder. In fact if K has an equation of the form $y = \phi(x)$ near $x = 0$, and if $\phi(x)$ possesses a derivative of any order which does not vanish at (0, 0), then z is positive along K except at (0, 0), even when $\phi(x)$ is non-analytic.

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[*] I shall understand by an analytic curve a curve which at each point may be represented by an equation of the form $y = \phi(x)$ [or $x = \psi(y)$], where $\phi(x)$ [or $\psi(y)$] is expansible by Taylor's infinite series at that point. This is easily shown to be equivalent to a definition in parameter form; compare e. g., a paper by Ames, *American Journal of Math.*, vol. 27 (1905), p. 343.

[†] See, e. g., Goursat, l. c., p. 106.

The original statement concerning the surface (2) is thus proved. It may be noted that each of the partial derivatives of z (of all orders) vanishes at (0, 0) except $\partial^2 z/\partial y^2$, which is identically equal to 2. It is evident, therefore, that none of the usual tests by means of partial derivatives would lead to any conclusion in this instance; for

$$B^2 - AC = z_{xy}^2 - z_{xx}\,z_{yy} = 0$$

at (0, 0), and all derivatives of order higher than the second vanish. It is also evident that z is not expansible in Taylor's double infinite series, for the sum of the series obtained by attempted expansion is y^2, which is equal to z only for $x = 0$, $y = 0$.

Models of this surface and of that of Peano are readily constructed in plaster; these at once allay the natural incredulity which at first attaches to such paradoxes.

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